Tatami Maker: A combinatorially rich mechanical game board

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Abstract

Japanese tatami mats are often arranged so that no four mats meet. This local restriction imposes a rich combinatorial structure when applied to monomino-domino coverings of rectilinear grids. We describe a modular, mechanical game board, prototyped with a desktop 3D printer, that enforces this restriction, and transforms tatami pen-and-paper puzzles into interactive sculptures. We review some recent mathematical discoveries on tatami coverings and present five new combinatorial games implemented on the game board.

Introduction

Tatami mats are a common floor furnishing, originating in aristocratic Japan, during the Heian period (794-1185). These thick mats, once hand-made with a rice straw core and a soft, woven rush straw exterior, are now machine-produced in a variety of materials, and are available in mass-market stores. They are so integral to Japanese culture, that a standard sized mat is the unit of measurement in many architectural applications (see [7]).

In our mathematical musings we depart considerably from traditional layouts, but we retain two essential items. The first of these is aspect ratio; a full mat is a 1×2 domino, and a half mat is 1×1 monomino. The second is the 17th century rule for creating auspicious arrangements; *no four mats may meet*.

Counting domino coverings is a classic area of enumerative combinatorics and theoretical computer science, but little attention has been paid to problems where the local interactions of the dominoes are resctricted in some fashion. The tatami restriction is perhaps the most natural of these, and it imposes a visually appealing structure with nice combinatorial properties (see Fig. 1). As a result, it is the subject of an exercise in Volume 4 of "The Art of Computer Programming" ([8]), where Knuth reprints a diagram of a 17th century Japanese mathematician, and recently the tatami restriction has been studied in several research papers (see [1, 5, 6, 9, 3, 4]).

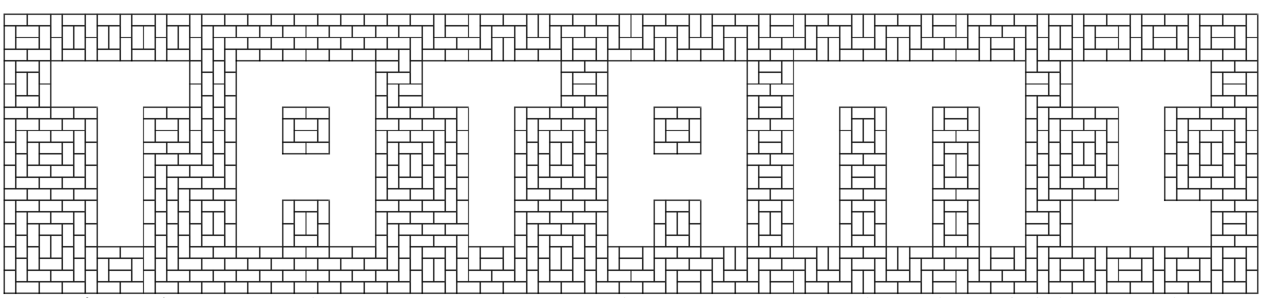


Figure 1 : *A tatami domino covering, generated in software using a logical satisfiability formula.*

Mathematics in art is an ideal pedagogic formula; the math challenges the logical capabilities of the participant, whereas the art compels her to understand it. The combination is even more effective for an interactive art installation that changes and increases gradually in sophistication as the participant furthers her engagement with it. An example of this is an installation called *Boundary Functions*, by Scott Snibbe (see [10]). Participants on the floor of the installation become points in a Voronoi diagram that is projected

onto the floor from the ceiling. That is, lines are projected to form a 2-dimensional cell around each participant. As more participants enter the installation and move around, the installation reveals more about the nature of Voronoi diagrams. Boundary Functions may or may not have been intended as a teaching tool, but learning would appear to be an inevitable consequence.

Snibbe's installation is inviting, aesthetically appealing, and self explanatory. I struggle to envision a Voronoi-themed pen-and-paper puzzle that qualifies as art, precisely because it would be missing these things. The same challenge presents itself for tatami coverings. We describe four tatami puzzle games which have the potential to become art by means of a kinetic sculpture.

On paper, a tatami puzzle is comprised of a grid and some instructions. The player must internalize the tatami restriction, and apply it every time she draws a tile. The structure of a tatami covering is not obvious to the uninformed, and applying the local restriction can overwhelm the player's efforts to solve the puzzle. Furthermore, the appearance of the pen-and-paper puzzle depends on the player's artistic ability and care, which can negate the natural beauty of tatami coverings.

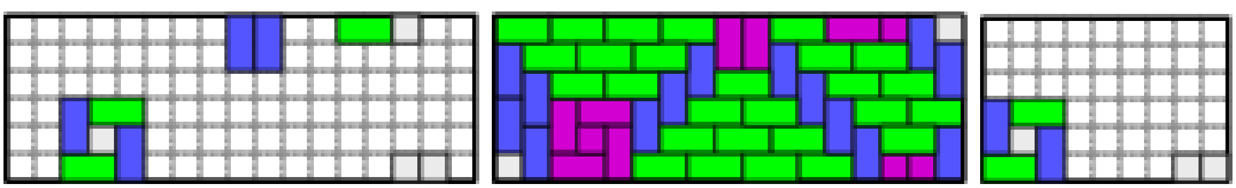
A mechanical game board that enforces the tatami restriction eliminates the preoccupation with "no four tiles meet". The uninformed player engages with the installation by trial and error and the sculpture teaches the tatami restriction incrementally. At the end she is rewarded with a completed tatami covering.

We describe the structure of tatami coverings and a prototypical kinetic sculpture called Tatami Maker, which realizes all of the combinatorial properties of tatami coverings. We introduce four puzzle games that are playable on Tatami Maker, along with some related mathematical results.

Structure

We introduce the tatami structure with an interactive demonstration. Reader, take up your pencil and complete the partial covering in Fig. 2(a). Once again, no four mats may meet at any point; alternatively, every intersection of the grid will contact a broad edge of a domino. You should find the completion is unique and equivalent to Fig. 2(b).

Completing the exercise will inevitably bring about the discovery of the *ray*, which occurs wherever a vertical domino shares an edge with a horizontal domino. The tatami restriction forces the pattern to repeat itself until it reaches the boundary of the grid.



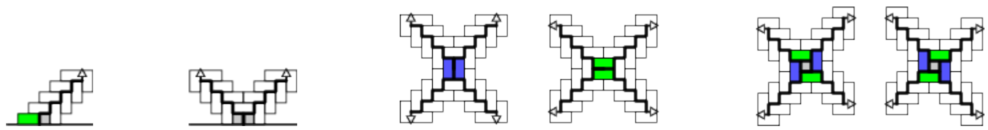
(a) (b) (c)

Figure 2: *Attempt to complete the partial tatami coverings by placing tiles which are forced by the tatami restriction.* (a)*, A partial covering and,* (b)*, its unique completion.* (b)*, This covering contains every possible type of feature.* (c) *This partial covering cannot be completed.*

The tiles in the partial covering in Fig. 2(c) are incompatible because the tatami restriction forces the propagation of rays that cross paths. I invite the reader to check.

Tatami coverings have the remarkable property that a simple local restriction imposes an aesthetically appealing, combinatorial structure. We discovered the ray in Fig. 2, and a case analysis of how rays begin reveals that in a rectangular covering, there are only four distinct ray-producing *features*, up to reflection and rotation. These four features and their rays are shown in Fig. 3 (see [5]). A brick laying pattern, called *bond*,

fills the areas between the features and rays. There are two types of horizontal bond and two types of vertical bond.



(a) loner (b) vee (c) bidimers (d) vortices

Figure 3 : *Tatami coverings of rectangular grids are comprised of these four types of feature and the bond pattern.*

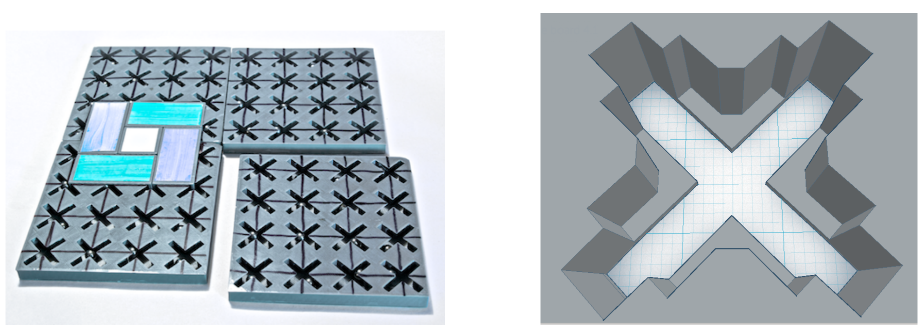
Likely the following theorem holds for general rectilinear regions, but it has only been proved for rectangular grids.

Lemma 1 ([5])**.** *Let G be an $r \times c$ grid, with $r < c$. A tatami covering of G is uniquely determined by the tiles touching its boundary.*

In terms of the area of the region, tatami coverings are simpler than general monomino-domino coverings by an entire order of complexity. The result is that a randomly chosen tatami covering has an inevitable simplicity that is beautiful, not only in its construction, but also in its appearance. In spite of this, the structure described here is not obvious to the uninformed, and building an intuition for it can take considerable effort.

Tatami Maker: a combinatorially rich mechanical game board

The Tatami Maker game board forms a rectilinear grid that enforces the tatami restriction when it is covered by the accompanying tile pieces. Arbitrary rectilinear grids can be created by placing Tatami Maker's modules alongside each other. A simple mechanism is embedded at each grid line intersection, which obstructs the placement of a tile if the other three incident grid squares are covered.



(a) Tatami Maker modules are placed alongside each other to create larger grids.

(b) The inside of Tatami Maker's intersection mechanism. Each arm of the `X` shape extends to a different grid square.

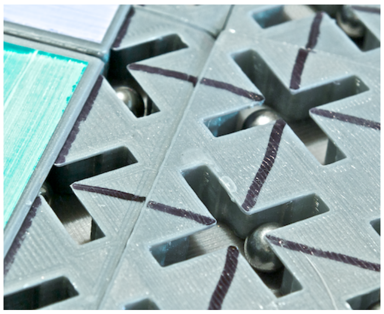
Figure 4 : *Tatami Maker modules and mechanism.*

A tile piece covers one or two grid squares, and the underside of each of its four corners has the specially shaped foot shown in Fig. 5(b). The feet interact with the mechanism in the game board by pushing an obstructing ball onto an unoccupied grid square, as well as guiding the tile correctly onto the grid.

Each mechanism occupies one grid intersection, which consists of one quadrant from each of four incident grid squares. A cavity in the game board contains a ball which may travel freely to any unoccupied

quadrant. If a quadrant is occupied by a tile's corner, then one of tile's feet occupies the part of the cavity that is otherwise available to the ball. When three of the quadrants are occupied by tile corners, the ball is forced to occupy the remaining quadrant, thereby preventing a tile corner from being placed here.

A minimal game board module is one grid intersection. As a result, the dimensions of a module are given in terms of its intersections rather than its grid squares. Our prototype's modules are 4×4, but they need not be square, or even rectangular.


(a)

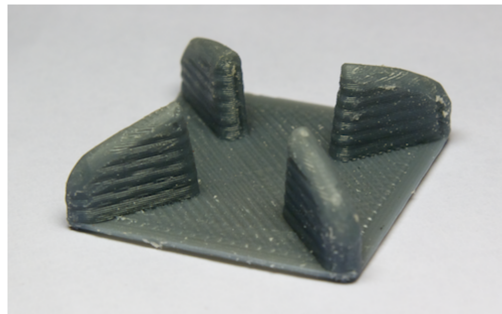

(b)

Figure 5 : (a), *Grid intersections and several Tatami Maker modules.* (b) *The feet of a tile piece.*

The main design challenge is ensuring that the ball can be pushed by an incoming foot, unobstructed, to an available quadrant. We label the quadrants Q1, Q2, Q3, and Q4, in counterclockwise order. A critical case occurs when Q2 and Q4 are occupied by tile corners (and feet), and a tile foot placed in Q1 must push the ball to Q3. Intuitively, the ball must disturb the feet in Q2 and Q4 on the way to Q3, otherwise the midpoint of the intersection would accomodate the ball when all four quadrants are occupied. The mechanism is designed, therefore, so that the ball lifts the tiles in Q2 and Q4 as it passes to Q3, and so that it will not become stuck against another part of the mechanism before it arrives in Q3.

The availability of desktop 3D printers has lowered the cost of creating prototypes sufficiently that rough estimation, and iterative trial and error was the most economical way of solving these design challenges. Tatami Maker was prototyped with a Solidoodle 2 printer; a 3D CNC machine, that extrudes a filament of hot ABS plastic into a $15 \times 15 \times 15$ cm print area to create real life versions of virtual 3D models. With each iteration of the prototype, changes to the shape of the cavity and the feet of the tile pieces were made to tease out the required behaviour of the mechanism.

Tatami Maker was tested informally at Science World in Vancouver, and it mesmerized many passing young children (see Fig. 6).


(a)


(b)

Figure 6 : *Visitors to Science World in Vancouver, Canada, test Tatami Maker by playing Oku.*

Five tatami puzzles

We describe how to set up five original combinatorial puzzle games for Tatami Maker, and highlight some related mathematical discoveries. The first four are for a single player, and the last one is for two players.

Oku From the Japanese word for "put", the most straightforward puzzle requires that the player cover a rectilinear region with tiles. The informed player can do this very easily by using a bond pattern, and placing monominoes wherever dominoes do not fit, but otherwise the pitfalls arising from the structure of tatami coverings are numerous (see Fig. 7), and Oku is particularly instructive in this regard.

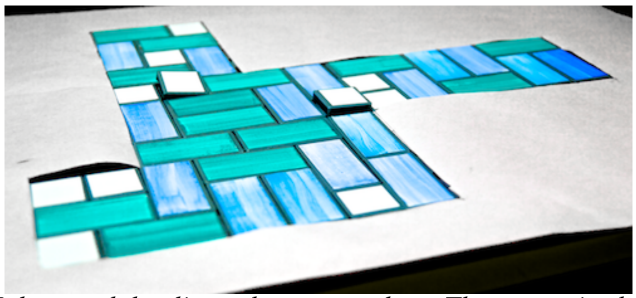


Figure 7 : *Tatami Maker modules lie under an overlay. The two raised monominoes are obstructed by Tatami Maker's mechanism.*

Some of the nicest combinatorial results for Oku are on square grids. The numbers of coverings for various parameters have simple representations, and divide up into natural equivalence classes (see [6]).

Theorem 1 ([5, 6]). *If n and m have the same parity, then the number of tatami coverings of the $n \times n$ grid with m monominoes is*

- $m2^m + (m+1)2^{m+1}$ *when* $m < n$,
- $n2^{n-1}$ *when* $m = n$; *and*
- 0 *when* $m > n$.

We can also generate certain subsets of these coverings, for example those with $m = n$ and a given number of vertical dominoes (see Fig. 8).

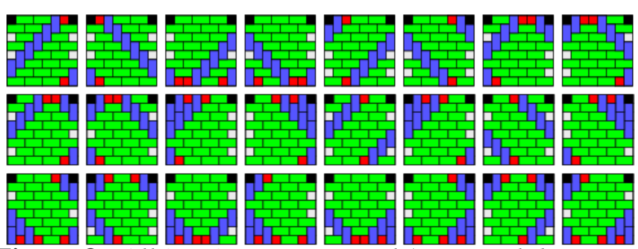
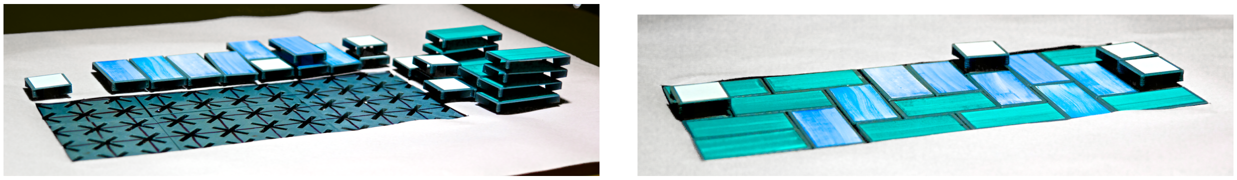


Figure 8 : *All 8×8 coverings with 7 vertical dominoes.*

Tomoku Due to Martín Matamala (private communication), Tomoku is a contraction of the words tomography, and the above game, Oku. This puzzle is played on a rectangular board, and the player is given the row and column projection of a set of solutions, one of which the player must find to complete the puzzle.

Tomoku is defined as a decision problem, where one must determine whether or not a solution exists. In practice, it is more interesting to reconstruct a covering when the answer is yes, than it is to solve a decision problem.



(a) (b)

Figure 9: (a), *Typical setup for an instance of Tomoku. Monominoes are double stacked in solution,* (b) *because they appear in both a column and a row.*

Problem 1. *Given an $r \times c$ grid, and a triple of integers, (v, h, m), for each row and each column, denoting the number of grid squares covered by vertical dominoes, horizontal dominoes, and monominoes, determine whether or not a covering exists with these row and column projections.*

It is not known whether a polynomial algorithm exists to solve an instance of Problem 1, but there are pairs of coverings with the same row and column projection. For example, an $n \times n$ covering with a central clockwise vortex gives the same row and column projections as a counterclockwise vortex.

An instance of Tomoku is set up by arranging Tatami Maker modules under an overlay which is printed with instructions and masks the modules to expose only the required grid. The instructions indicate how to set up the modules and overlay, as well as the details of one or more instances of Tomoku (see Fig. 10).

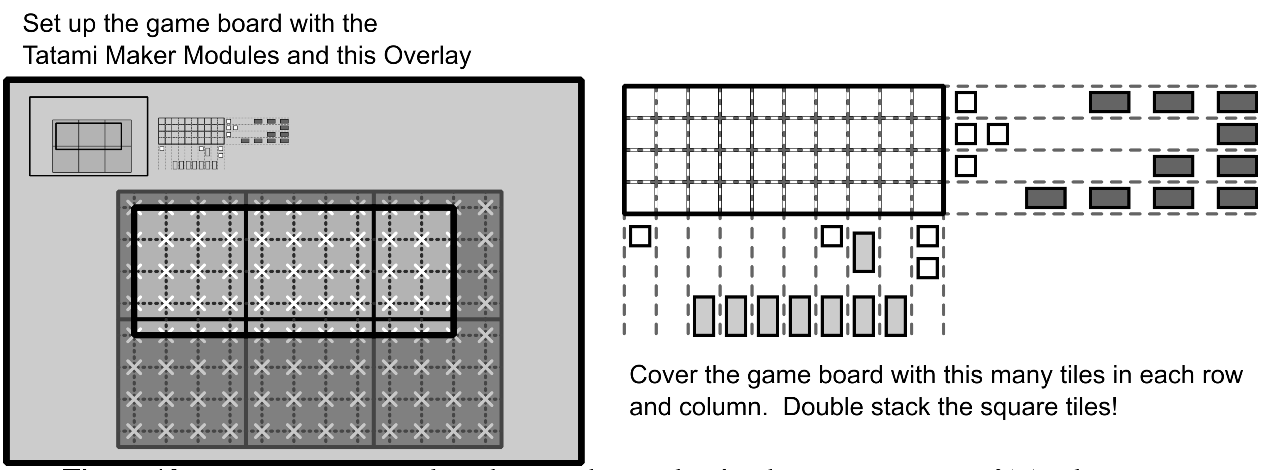


Figure 10: *Instructions printed on the Tomoku overlay for the instance in Fig. 9(a). This particular one can be completed without backtracking.*

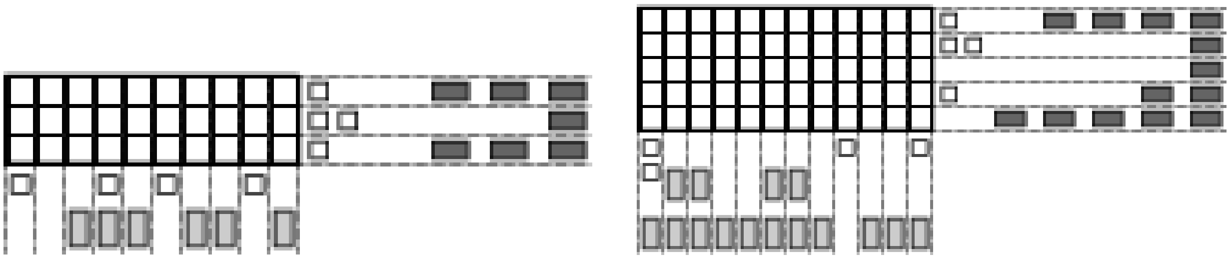
Tomoku may be played with pencil and paper, and two instances are reprinted in Fig. 11 from [2]. Considerable efficiency can be acheived by using short line segments to represent dominoes, and dots to represent monominoes.

The Lazy Paver The Lazy Paver is tasked with tatami covering a rectilinear driveway with pavers. She would typically produce a 1×1 paver by cutting a 1×2 paver in half, so instead, she will avoid the extra work by covering the driveway with domino pavers.

The Lazy Paver is also defined as a decision problem.

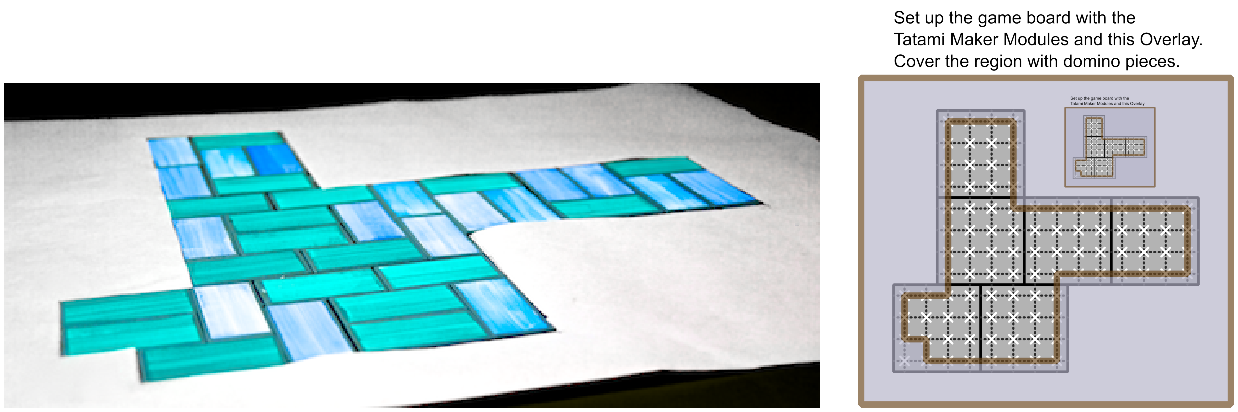
Problem 2 ([5])**.** *Determine whether or not a given rectilinear region can be tatami covered with dominoes.*

The Lazy Paver problem is known to be NP-complete ([4]). That is, Problem 2 is as difficult to solve,



(a) (b)

Figure 11 : *Instances of Tomoku, reprinted from [2]. The 5×12 puzzle is quite challenging.*



(a) (b)

Figure 12 : *Lazy Paver instance and instructions printed on an overlay.*

in terms of the number of corners in the input region, as the most difficult decision problems whose answers can be checked in polynomial time.

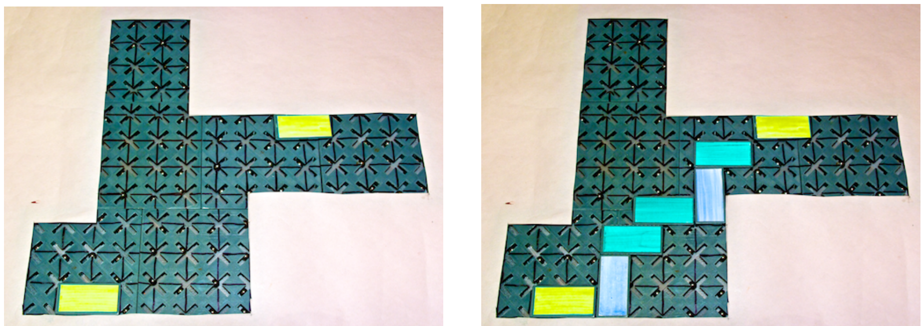
Once again, we set up the game as a covering problem, rather than a decision problem, by placing an overlay on top of the Tatami Maker modules. The overlay has a rectilinear region cut out of it, as well as the instruction to cover the region with dominoes. A prototypical version and instructions appear in Fig. 12.

The Paving Consultant Dr. Matt DeVos communicated this final decision problem to me, and it is also playable as a covering game (private communication).

Problem 3. *Determine whether a partial covering of a given rectilinear region can be completed.*

A Paver, perhaps a lazy one, has abandoned her job, and left a partially paved region. The Paving Consultant must decide whether the covering can be completed. The puzzle can be set up with overlays, as above, bearing a diagram of the partial covering and indications on how to arrange the Tatami Maker Modules. Tiles of a different colour can be provided so that the player does not mix these up with the tiles that are part of the solution (see Fig. 13).

Noku Noku is an adversarial game, perhaps antithetical to Oku, proposed by Frank Ruskey (private communication). Players alternately place tiles onto the game board in order to win by forcing a position where their opponent has no available move. A computer analysis of some of Noku's smallest game trees shows that Player 2 can force a win on the 2×6 grid, which has 431949 nodes, but the first player wins for other similarly small cases, including 4×3 and 4×4.



(a) (b)

Figure 13 : (a), *A NO instance of the Paving Consultant. Placing more tiles,* (b), *hints at why this is so, and suggests an alternate definition for the tatami restriction.*

Future

I hope to produce more polished prototypes and test the playability of these puzzles thoroughly. It is unclear whether Tatami Maker is suited to becoming a larger art installation, or a boxed puzzle set, but it has already given tangible life to an otherwise abstract research topic.

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